JOURNAL OF COMBINATORIAL THEORY, Series B 24, 258–266 (1978)

A Minimum-Vertex Triangulation

JOHN PHILIP HUNEKE

The Ohio State University, Columbus, Ohio 43210

Communicated by the Managing Editors

Received December 19, 1975

An Euler characteristic argument indicates that if K, a girth three graph, triangulates the genus two orientable surface, then K contains at least nine vertices. A proof is given that such a graph K must contain at least 10 vertices.

1. INTRODUCTION

Two underlying propositions, one by Euler and the second a combinatorial observation, can be blamed for a number of conjectures concerning graphs to be embedded or immersed in certain surfaces (i.e., two-dimensional compact topological manifolds). Suppose a connected graph K, containing a cycle, with m edges, n vertices, and girth p is embedded into an orientable surface, Σ_g, of genus g, or into a nonorientable surface $\tilde{\Sigma}_{\tilde{g}}$ of genus $\tilde{g}$, making r regions (i.e., the complement of K in Σ_g, or respectively $\tilde{\Sigma}_{\tilde{g}}$, has r components) Then:

PROPOSITION 1.1 (Euler).

(i) $n - m + r \geqslant 2 - 2g$;

(ii) $n - m + r \geqslant 2 - \tilde{g}$; *and*

(iii) *in* (i), *respectively* (ii), *equality holds provided each of the r regions is contractible* (*i.e., homeomorphic to the real plane*).

PROPOSITION 1.2. *$p \cdot r \leqslant 2m$ with equality provided the boundary of each region contains exactly p edges of K.*

Proof. Each region has at least p edges in its boundary and each edge is in the boundary of at most two regions.

COROLLARY 1.3.

(i) $g \geqslant 1 + m/2 - m/p - n/2$;

(ii) $\tilde{g} \geqslant 2 + m - 2m/p - n$; *and*

0095-8956/78/0243-0258$02.00/0

(iii) *in* (i), *respectively* (ii), *equality holds provided each region is contractible and is bounded by exactly p edges.*

Observe that Corollary 1.3 gives a condition on the surface in terms of invariants of the graph K which is necessary to have an embedding; occasionally it is also a sufficient condition to ensure the existence of an embedding. For example,

COROLLARY 1.4. *Let K be the complete graph on n vertices embedded in Σ_g and $\tilde{\Sigma}_{\tilde{g}}$. Then:*

(i) $g \geqslant (n-3)(n-4)/12$;

(ii) $\tilde{g} \geqslant (n-3)(n-4)/6$; *and*

(iii) *the embedding is a triangulation provided equality holds in* (i), *respectively* (ii).

Proof. $K = K_n$ implies $p = 3$ and the number of edges of K_n is $m = n(n-1)/2$.

Ringel and Youngs [5] established the Heawood conjecture [2] by establishing that the necessary relation between n and g, or $\tilde{g}$, given in Corollary 1.4 for K_n to embed in Σ_g, or in $\tilde{\Sigma}_{\tilde{g}}$, is sufficient with the single exception of $n = 7$ and $\tilde{g} = 2$. In other words,

THEOREM 1.5 [4]. (i) *The smallest genus g such that K_n embeds in Σ_g is the integer hull of* $(n-3)(n-4)/12$, *and*

(ii) *the smallest genus $\tilde{g}$ such that K_n embeds in $\tilde{\Sigma}_{\tilde{g}}$ is the integer hull of* $(n-3)(n-4)/6$ *provided $n \neq 7$ (K_7 embeds in $\tilde{\Sigma}_3$ but not $\tilde{\Sigma}_2$.)*

Observe by Corollary 1.4 and Theorem 1.5 that K_n triangulates Σ_g if and only if $g = (n-3)(n-4)/12$, and K triangulates $\tilde{\Sigma}_{\tilde{g}}$ if and only if $2 \neq \tilde{g} = (n-3)(n-4)/6$. Also, if $g > (n-3)(n-4)/12$, K_n embeds in Σ_g but not as a triangulation, and if $\tilde{g} > (n-3)(n-4)/6$ then K_n embeds in $\tilde{\Sigma}_{\tilde{g}}$ but not as a triangulation. Hence, a necessary condition for a subgraph of K to triangulate Σ_g is for $g \leqslant (n-3)(n-4)/12$, and for K to triangulate $\tilde{\Sigma}_{\tilde{g}}$ is for $\tilde{g} \leqslant (n-3)(n-4)/6$. Hence, the conjecture analogous to Theorem 1.5:

Conjecture 1.6. (i) *The smallest n such that Σ_g can be triangulated by a subgraph of K is the smallest n such that g is the integer part of* $(n-3)\cdot(n-4)/12$, *and*

(ii) *the smallest n such that $\tilde{\Sigma}_{\tilde{g}}$ can be triangulated by a subgraph of K_n is the smallest n such that $\tilde{g}$ is the integer part of* $(n-3)(n-4)/6$, *provided $n \neq 7$.*

Ringel [6] established another (besides $n = 7$) exemption to the nonorientable half of Conjecture 1.6 by proving that $n = 8$, $\tilde{g} = 3$ does not satisfy Conjecture 1.6(ii).

Kainen has a crossing number conjecture [3] which also can be derived from Corollary 1.3(i) above, and which would imply Conjecture 1.6(i) above; Kainen verified his conjecture for $n < 9$.

The purpose of this note is to establish one counterexample to Conjecture 1.6(i) above (hence also to Kainen's conjecture.) However, it seems to the author that this exception to the conjecture could be a low genus anomaly analogous to the $n = 7$ exempted from Theorem 1.5(ii). This note exempts $n = 9$ (and $g = 2$) from Conjecture 1.6(i) with a proof of

THEOREM 1.7. *No subgraph of* K_9, *the complete-9 graph, triangulates* Σ_2, *the genus 2 orientable surface.*

Proof. By Corollary 1.3(iii) any subgraph of K_9 which triangulates Σ_2 must contain 33 edges. No subgraph of K_9 with 33 edges embeds on Σ_2 by Theorem 3.3 below. Hence the result.

Furthermore, a subgraph of K_{10} does triangulate Σ_2 as evidenced by Fig. 1.

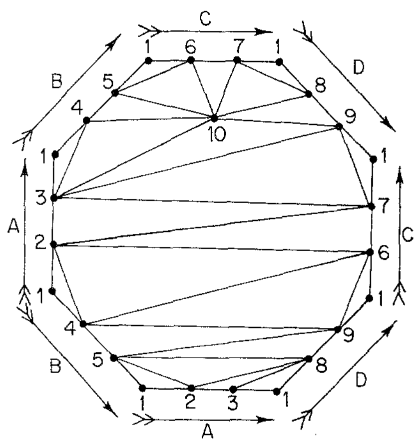

FIGURE 1

2. FUNDAMENTALS

Notation. (i) Σ_2 denotes the orientable genus 2 surface with an orientation presumed to be fixed unless specifically reversed.

(ii) K_9 denotes the complete 9 graph.

(iii) $\mathbb{Z}_n = \{i \mid i \text{ is an integer } 0 \leqslant i < n\}$ with the cyclic mod n additive group structure, for each natural number n.

(iv) An edge of K_9 with end points u and v is denoted (u, v); hence $(u, v) = (v, u)$.

(v) An oriented cycle in K_9 composed of three edges (u, v), (v, w), and (w, u) is denoted (u, v, w); hence $(u, v, w) = (v, w, u) = (w, u, v)$, but $(u, v, w) \neq (w, v, u)$ as these two have opposite orientations.

Standing Hypothesis. Assume throughout the remainder of this note that K is a subgraph of K_9 with exactly 33 distinct edges, and assume Δ: $K \to \Sigma_2$ is an embedding (observe that Δ triangulates Σ_2 by Corollary 1.3).

Notation (continued). (vi) If $C = (u, v, w)$ and if $\Delta(C)$ is the boundary of one component of $\Sigma_2 - \Delta(K)$, and if the orientation on $\Delta(C)$ induced by Δ from the orientation on C is compatible with the orientation of Σ_2, then we say (u, v, w) is a Δ-boundary.

PROPOSITION 2.1. *$K_9 - K$ is one of the five graphs without loops or double edges containing exactly three edges (see Fig. 2), so K has a valency sequence of* (8, 8, 8, 7, 7, 7, 7, 7, 7), (8, 8, 8, 8, 7, 7, 7, 7, 6), (8, 8, 8, 8, 8, 7, 7, 7, 5), (8, 8, 8, 8, 8, 7, 7, 6, 6), *or* (8, 8, 8, 8, 8, 8, 6, 6, 6).

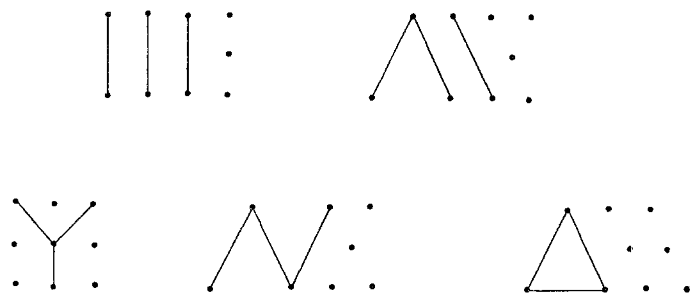

FIGURE 2

Proof. K_3 has $9 \cdot 8/2 = 36$ edges. All graphs with three edges are listed by Harary [1].

PROPOSITION 2.2. *For each vertex v with valency k in K, there are k distinct vertices, denoted v_i, $i \in \mathbb{Z}_k$, such that $\{(v_i, v, v_{i+1}) \mid i \in \mathbb{Z}_k\}$ is the set of all Δ-boundaries containing v, and $k \geqslant 5$.*

Proof. The set of Δ-boundaries containing v can be arranged cyclically because Σ_2 is a manifold, so a neighborhood of $\Delta(v)$ is homeomorphic to the real plane. The k vertices are distinct because $K \subset K_9$ and K_9 has girth 3. Proposition 2.1 implies $k \geqslant 5$.

COROLLARY 2.3. *If v is a vertex of K, $\mathscr{B} = \{(v, v_0{}^i, v_1{}^i) \mid i \in \mathbb{Z}_6\}$ is a set of six distinct Δ-boundaries, and $\{v_j{}^i \mid i \in \mathbb{Z}_6, j \in \mathbb{Z}_2\}$ is a set of eight distinct vertices, then all eight Δ-boundaries containing v are uniquely determined by*

Proposition 2.2 *to be either in* $\mathscr{B}$ *or of the form* (v, v_1^i, v_0^j) *where* $i \neq j$ *and the vertices* v_1^i *and* v_0^j *each is in only one* Δ*-boundary in* $\mathscr{B}$.

3. Proof of Main Theorem

This section uses the notation and propositions of Section 2 to establish Theorem 3.3. Recall the

Standing Hypothesis. Assume K is a subgraph of K_9 with 33 edges and $\Delta: K \to \Sigma_2$ is an embedding, which triangulates Σ_2.

Lemma 3.1. *For each vertex valency* 8, *denoted* ∞, *there is a labeling of the remaining vertices by* $\mathbb{Z}_8$ *with* $(i, i+1, \infty)$ *a* Δ*-boundary for each* $i \in \mathbb{Z}_8$ *such that* $(1, 0, 3)$ *is a* Δ*-boundary* (*for some orientation of* Σ_2.)

Proof. Assume there is a valency 8 vertex, ∞, and assume $(1, 0, 3)$ is not a Δ-boundary for each labeling of the vertices such that $(i, i+1, \infty)$ is a Δ-boundary for each $i \in \mathbb{Z}_8$. Let $f: \mathbb{Z}_8 \to \mathbb{Z}_8$ be defined such that $(i+1, i, f(i))$ is a Δ-boundary. For each $i \in \mathbb{Z}_8$, $f(i) \notin \{i+2, i+1, i, i-1\}$ since otherwise vertices i or $i+1$ would have valency at most 3 by Proposition 2.2. For each $i \in \mathbb{Z}_8$, $f(i) \notin \{i+3, i-2\}$ since otherwise relabeling the vertices by changing j to $(j-i)$ for each $j \in \mathbb{Z}_8$, or by changing j to $(i-j+1)$ for each $j \in \mathbb{Z}_8$ (and reversing the orientation of Σ_2), yields a new labeling with Δ-boundaries $(1, 0, 3)$ and $(j, j+1, \infty)$ for all $j \in \mathbb{Z}_8$. Hence $f(i) \in \{i+4, i+5\}$ for each $i \in \mathbb{Z}_8$. Without loss of generality, assume $f(0) = 5$. Inductively, $f(i) = i+5$ for each $i \in \mathbb{Z}_8$ since if $f(k) = k+5$ and $f(k+1) \neq (k+1)+5$ then $f(k+1) = k+5$, so by Proposition 2.2, vertex $k+1$ would have valency 4, a contradiction. Now $\Delta|_{K'}$, for K' a subgraph of K_9 with 28 edges, has been established with at least 16 triangular regions and the remaining region or regions bounded by $\{\Delta(E), E$ an edge in one but not two of the 16 established Δ-boundaries$\}$; since these edges form a simple cycle (see Fig. 3) $\Delta|_{K'}: K' \to \Sigma_2$ has exactly 17 regions, each of which must

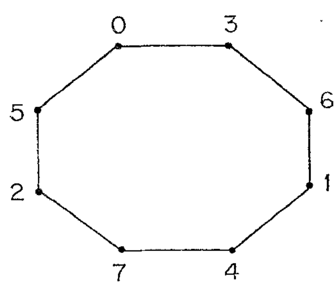

Figure 3

be contractible by Euler (i.e., Proposition 1.1). Hence, it remains to analyze how Δ could embed 5 of the 8 remaining edges $\{(i, i+2) \mid i \in \mathbb{Z}_8\}$ into the nontriangular region A (depicted in Fig. 3 with $\Delta(i)$ labeled by i).

Observe that at least one of the four edges $(i, i+2)$ with i odd (each of which is not in K') is in K. For each i odd, if $(i, i+2)$ is in K then $\Delta(i, i+2) \subset A$ so $(i, i+2, i+5)$ is a Δ-boundary and hence $i+5$ has valency 5 by Proposition 2.2. If only one of the edges of the form $(i, i+2)$ with i odd is in K then $K_9 - K$ contains at least three edges containing $(i+5)$ and two edges containing $i+4$, contradicting Proposition 2.1. However, if more than one of the edges of the form $(i, i+2)$ with i odd is in K then there is more than one vertex valency 5, again contradicting Proposition 2.1. Hence the result.

Lemma 3.2. *There is a vertex valency* 8, *denoted* ∞, *and there is a labeling of the remaining* 8 *vertices by* $\mathbb{Z}_8$ *such that for each* $i \in \mathbb{Z}_8$, $(i, i+1, \infty)$, *and* $(1, 0, 3)$, $(3, 0, 5)$, *and* $(4, 3, 7)$, *are* Δ-*boundaries* (*for some orientation of* Σ_2).

Proof. Label vertices $\{\infty\} \cup \mathbb{Z}_8$ such that $(1, 0, 3)$ and $(i, i+1, \infty)$ are Δ-boundaries for each $i \in \mathbb{Z}_8$; this is possible by Lemma 3.1. Denote by x and y the vertices such that $(3, 0, x)$ and $(4, 3, y)$ are Δ-boundaries. If $x = 1$ then since $(1, 0, 3)$ is a Δ-boundary, 0 has valency 2 by Proposition 2.2, a contradiction. If $x = 2$ then $(3, 0, x) = (2, 3, 0)$ and $(2, 3, \infty)$ is a Δ-boundary, contradicting Proposition 2.2. If $x = 7$ then by Proposition 2.2, vertex 0 has valency 4, a contradiction. Thus $x \in \{4, 5, 6\}$. If $y = 5$ then by Proposition 2.2 vertex 4 has valency 3, a contradiction. If $y = 1$ then $(4, 3, 1)$, $(1, 0, 3)$ are Δ-boundaries, contradicting Proposition 2.2. If $y = 2$ then $(4, 3, 2)$ is a Δ-boundary so by Proposition 2.2, vertex 3 has valency 3, a contradiction. Thus $y \in \{6, 7, 0\}$; also $y = 0$ if and only if $x = 4$ as $(4, 3, 0) = (3, 0, 4)$. The remainder of the proof that $x = 5$, $y = 7$ will be to successively eliminate three other cases: Case 1, $x = 4, y = 0$; Case 2, $x = y = 6$; and Case 3, $x + 1 = y \in \{6, 7\}$.

Case 1 ($x = 4$, $y = 0$) *assumption*: for each orientation of Σ_2, and for each labeling of the vertices of K, if vertex ∞ has valency 8, and for each $i \in \mathbb{Z}_8$ $(i, i+1, \infty)$, and $(1, 0, 3)$ are Δ-boundaries, then also $(3, 0, 4)$ is a Δ-boundary. Assume we have a labeling satisfying the hypothesis of the Case 1 assumption. Consider $(0, 7, z)$. If $z = 1$ (or $z = 6$) then vertex 0 (or 7 respectively) is valency 3 by Proposition 2.2, a contradiction. If $z = 3$, then $(0, 7, 3)$ and $(3, 0, 4)$ are Δ-boundaries contradicting Proposition 2.2. If $z = 4$, then vertex 0 has valency 5 by Proposition 2.2, so vertex 4 has valency 8 by Proposition 2.2; relabeling, changing 4 to ∞, 7 to 0, 0 to 1, 3 to 2, ∞ to 3, 5 to 4, and the other vertices appropriately so that under the new labeling $(i+1, i, \infty)$ is a Δ-boundary, we have (reversing the orientation on Σ_2) for each i, $(i, i+1, \infty)$ and $(1, 0, 3)$ are Δ-boundaries but the Case 1 hypothesis implies $(3, 0, 4)$ is a Δ-boundary which is $(\infty, 7, 5)$ in the original notation, a contradiction by Proposition 2.2. If $z = 5$, then relabeling, changing i to $-i$ for each $i \in \mathbb{Z}_8$ (and reversing the orientation of Σ_2) we have with the new labeling $i \in \mathbb{Z}_8$ $(i, i+1, \infty)$, $(1, 0, 3)$, and $(5, 4, 0)$ are Δ-boundaries and by the Case 1 assumption, $(3, 0, 4)$ is a Δ-boundary so by Proposition 2.2, vertex 4 has valency 4, a contradiction. Hence z must be vertex 2. Specifically, we have under the Case 1 assumption, for any labeling of the vertices of K if vertex ∞ has valency 8 and for each $i \in \mathbb{Z}_8$, $(i, i+1, \infty)$ is a Δ-boundary and for some j, $(j+1, j, j+3)$ is a Δ-boundary, then $(j+3, j, j+4)$ and $(j, j-1, j+2)$ are Δ-boundaries. However, this is an inductive property on j so for the assumed labeling we have $(j, j-1, j+2)$ a Δ-boundary for each $j \in \mathbb{Z}_8$. However, then $(3, 0, 4)$ and (for $j = 4$) $(4, 3, 6)$ are Δ-boundaries contradicting Proposition 2.2. Hence the Case 1 assumption leads to a contradiction.

Case 2 $(x = y = 6)$ *assumption*: for each labeling of the vertices of K if vertex ∞ has valency 8 and for each $i \in \mathbb{Z}_8$ $(i, i+1, \infty)$ and $(1, 0, 3)$ are Δ-boundaries, but $(3, 0, 4)$ is not a Δ-boundary then $(3, 0, 6)$ and $(4, 3, 6)$ are Δ-boundaries. By Case 1, we may suppose we have a labeling satisfying the hypothesis of the Case 2 assumption. Then relabel the vertices by changing i to $i-3$ for each $i \in \mathbb{Z}_8$. With the new notation we have $i \in \mathbb{Z}_8$ $(i, i+1, \infty)$, $(1, 0, 3)$, and $(0, 5, 3)$ are Δ-boundaries, hence neither $(3, 0, 4)$ nor $(3, 0, 6)$ is a Δ-boundary by Proposition 2.2, contradicting the Case 2 assumption. Hence Case 2 is not possible.

Case 3 $(x + 1 = y \in \{6, 7\})$ *assumption*: the vertices of K are labeled such that vertex ∞ has valency 8 and for each $i \in \mathbb{Z}_8$ $(i, i+1, \infty)$ $(1, 0, 3)$, $(3, 0, x)$, and $(4, 3, y)$ are Δ-boundaries where $x + 1 = y \in \{6, 7\}$. Let $z \in \mathbb{Z}_8$ such that $\{x, y, z\} = \{5, 6, 7\}$. First determine u such that $(3, 2, u)$ is a Δ-boundary. If $u \in \{1, 4, y, \infty\}$ then vertex 3 has valency less than 5 by Proposition 2.2, a contradiction. If $u \in \{x, 0\}$ then $(3, 0, x)$ and $(1, 0, 3)$ are Δ-boundaries contradicting Proposition 2.2. Hence $u = z$, so $(3, 2, z)$ is a Δ-boundary and vertex 3 has valency 8. Now, seek to determine w such that $(y, 3, w)$ is a Δ-boundary. If $w = z$, then vertex 3 has valency 5 by Proposition 2.2, a contradiction. If $w \in \{0, 1, 2, 4, \infty\}$ then $(3, 0, x)$, $(1, 0, 3)$, $(3, 2, z)$, $(3, 4, \infty)$, and $(2, 3, \infty)$ Δ-boundaries, contradicting Proposition 2.2. However, if $w = x$ then $(x, y, 3)$ and (x, y, ∞) are Δ-boundaries since $y = x + 1$, which contradicts Proposition 2.2. Hence no vertex is available for w, so edge $(y, 3)$ appears in only one Δ-boundary contradicting Proposition 2.2. Hence Case 3 is not possible.

Exclusion of Cases 1, 2, and 3 establishes Lemma 3.2.

THEOREM 3.3. *No subgraph of* K_9 *with* 33 *edges embeds in* Σ_2.

Proof. The standing hypothesis assures $\Delta: K \to \Sigma_2$ is an embedding for K a subgraph of K_9 with 33 edges. By Lemma 3.2, label the vertices of K such that $i \in \mathbb{Z}_8$, $(i, i+1, \infty)$, $(1, 0, 3)$, $(3, 0, 5)$ and $(4, 3, 7)$ are Δ-boundaries. We seek further necessary Δ-boundaries until a contradiction is established to the existance of Δ.

First, determine a such that $(3, 2, a)$ is a Δ-boundary. Since no new Δ-boundaries contain vertex ∞, $a \neq \infty$. Since $(1, 0, 3)$ is a Δ-boundary, $a \neq 0$. By Proposition 2.2, $a \neq 1$ or 4. Since $(3, 0, 5)$ is a Δ-boundary, $a \neq 5$ by Proposition 2.2. And since $(4, 3, 7)$, $(3, 4, \infty)$ and $(2, 3, \infty)$ are Δ-boundaries if $a = 7$, then vertex 3 has valency 4 by Proposition 2.2, a contradiction. Hence $a = 6$ and $(3, 2, 6)$ is a Δ-boundary, so vertex 3 has valency 8, and by Corollary 2.3, $(3, 6, 1)$ and $(3, 5, 7)$ are Δ-boundaries, and no more Δ-boundaries contain vertex 3.

Second, determine b such that $(0, 7, b)$ is a Δ-boundary. Since no undetermined Δ-boundaries contain vertices 3 or ∞, $b \neq 3$ or ∞. By Proposition 2.2, $b \neq 1$ or 6. Since $(4, 3, 7)$ is a Δ-boundary, $b \neq 4$ by Proposition 2.2. Since $(3, 5, 7)$ and $(3, 0, 5)$ are Δ-boundaries, if $b = 5$ then vertex 5 has valency 3 by Proposition 2.2, a contradiction. Hence $b = 2$ and $(0, 7, 2)$ is a Δ-boundary.

Third, determine c such that $(7, 6, c)$ is a Δ-boundary. Since no undetermined Δ-boundaries contain 3 or ∞, $c \neq 3$ or ∞. By Proposition 2.2, $c \neq 0$ or 5. Since $(3, 6, 1)$ is a Δ-boundary, $c \neq 1$. Since $(0, 7, 2)$, $(7, 0, \infty)$ and $(6, 7, \infty)$ are Δ-boundaries, if $c = 2$ then vertex 7 would have valency 4 by Proposition 2.2, a contradiction. Hence $c = 4$ and $(7, 6, 4)$ is a Δ-boundary.

Fourth, determine d such that $(6, 5, d)$ is a Δ-boundary. Since no undetermined Δ-boundaries contain 3 or ∞, $d \neq 3$ or ∞. By Proposition 2.2, $d \neq 7$ or 4. Since $(3, 2, 6)$ is a Δ-boundary, $d \neq 2$ by Proposition 2.2. If $d = 0$ then $(6, 5, 0)$, $(5, 6, \infty)$, $(6, 7, \infty)$, $(7, 6, 4)$, $(3, 2, 6)$ and $(3, 6, 1)$ are Δ-boundaries so by Corollary 2.3 $(6, 0, 1)$ is a Δ-boundary, which contradicts Proposition 2.2, since $(0, 1, \infty)$ is a Δ-boundary. Hence $d = 1$ and $(6, 5, 1)$ is a Δ-boundary.

Fifth, seek to determine e such that $(5, 4, e)$ is a Δ-boundary. Since no undetermined boundaries contain 3 or ∞, $e \neq 3$ or ∞. By Proposition 2.2, $e \neq 6$. Since $(3, 0, 5)$ is a Δ-boundary, $e \neq 0$ by Proposition 2.2. Since $(6, 5, 1)$, $(5, 6, \infty)$, and $(4, 5, \infty)$ are Δ-boundaries, if $e = 1$ then vertex 5 has valency 4 by Proposition 2.2, a contradiction. Since $(4, 3, 7)$, $(3, 4, \infty)$ and $(4, 5, \infty)$ are Δ-boundaries, if $e = 7$ then vertex 4 has valency 4 by Proposition 2.2, a contradiction. Also, if $e = 2$, then $(5, 4, 2)$, $(3, 5, 7)$, $(3, 0, 5)$, $(6, 5, 1)$, $(5, 6, \infty)$ and $(4, 5, \infty)$ are Δ-boundaries so by Corollary 2.3, $(5, 0, 1)$ is a Δ-boundary, which contradicts Proposition 2.2 since $(0, 1, \infty)$ is a Δ-boundary. Hence each vertex has been excluded as a possibility for e so $(4, 5, \infty)$ is the only Δ-boundary containing edge $(4, 5)$, contradicting Proposition 2.2. Hence the result.

REFERENCES

1. F. HARARY, "Graph Theory," Appendix 1, Addison–Wesley, Philippines, 1969.
2. P. J. HEAWOOD, Map colour theorem, *Quart. J. Math.* **24** (1890), 332–338.
3. P. KAINEN, A lower bound for crossing numbers of graphs with applications to K_n, $K_{p,q}$ and $Q(d)$, *J. Combinatorial Theory Ser. B* **12** (1972), 287–298.
4. G. RINGEL, "Map Color Theorem," Springer-Verlag, New York, 1974.
5. G. RINGEL AND J. W. T. YOUNGS, Solution of the Heawood map-coloring problem, *Proc. Nat. Acad. Sci. USA* **60** (1968), 438–445.
6. G. RINGEL, Wie man die geschlossenen nichtorientiebaren Flächen in moglichst wenig Dreiecke zerlegen kann, *Math. Ann.* **130** (1955), 317–326.